

IMPRIMITIVE PERMUTATIONS GROUPS GENERATED BY THE ROUND FUNCTIONS OF KEY-ALTERNATING BLOCK CIPHERS AND TRUNCATED DIFFERENTIAL CRYPTANALYSIS

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ABSTRACT. We answer a question of Paterson, showing that all block systems for the group generated by the round functions of a key-alternating block cipher are the translates of a linear subspace. Following up remarks of Paterson and Shamir, we exhibit a connection to truncated differential cryptanalysis.

We also give a condition that guarantees that the group generated by the round functions of a key-alternating block cipher is primitive. This applies in particular to AES.

1. INTRODUCTION

Kenneth Paterson [12] has considered iterated block ciphers in which the group generated by the one-round functions acts imprimitively on the message space, with the aim of exploring the possibility that this might lead to the design of trapdoors. The blocks of imprimitivity he uses are the translates (cosets) of a linear subspace. He asked whether it is possible to construct other, non-linear blocks of imprimitivity.

In the first part of this paper we answer this question in the negative for key-alternating block ciphers, and exhibit a connection to truncated differential cryptanalysis, following up remarks of Paterson and Shamir.

We then develop a conceptual recipe to guarantee that the group generated by the one-round functions of a key-alternating block cipher acts primitively on the message space. We show that the conditions we require are satisfied in a natural way by AES.

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2. PRELIMINARIES

Let G be a finite group, acting transitively on a set V . We write the action of an element $g \in G$ on an element $\alpha \in V$ *on the right*, that is, as αg . Also, $\alpha G = \{\alpha g : g \in G\}$ is the orbit of α under G , and $G_\alpha = \{g \in G : \alpha g = \alpha\}$ is the stabilizer of α in G .

A *partition* of V is a family \mathcal{B} of nonempty subsets of V such that any element of V lies in precisely one element of \mathcal{B} . A partition \mathcal{B} is said to be *G -invariant* if for any $B \in \mathcal{B}$ and $g \in G$, one has $Bg \in \mathcal{B}$. A G -invariant partition \mathcal{B} is said to be *trivial* if $\mathcal{B} = \{V\}$, or $\mathcal{B} = \{\{\alpha\} : \alpha \in V\}$.

A non-trivial, G -invariant partition of V is said to be a *block system* for the action of G on V . If such a block system exists, then we say that G is *imprimitive* in its action on V (equivalently, G *acts imprimitively* on V), *primitive* otherwise. An element B of some block system \mathcal{B} is called a *block*; since G acts transitively on V , we have then $\mathcal{B} = \{Bg : g \in G\}$.

We note the following elementary

Lemma 2.1 ([1], Theorem 1.7). *Let G be a finite group, acting transitively on a set V . Let $\alpha \in V$.*

Then the blocks B containing α are in one-to-one correspondence with the subgroups H , with $G_\alpha < H < G$. The correspondence is given by $B = \alpha H$.

In particular, G is primitive if and only if G_α is a maximal subgroup of G .

We will need a fact from the basic theory of finite fields. (See for instance [6] or [8].) Write $\mathbf{GF}(p^n)$ for the finite field with p^n elements, p a prime.

Lemma 2.2. $\mathbf{GF}(p^n) \subseteq \mathbf{GF}(p^m)$ if and only if n divides m .

In the rest of the paper, we tend to adopt the notation of [2].

Let $V = V(n_b, 2)$, the vector space of dimension n_b over the field $\mathbf{GF}(2)$ with two elements, be the state space. V has 2^{n_b} elements.

For any $v \in V$, consider the translation by v , that is the map

$$\begin{aligned} \sigma_v : V &\rightarrow V, \\ w &\mapsto w + v. \end{aligned}$$

In particular, σ_0 is the identity map on V . The set

$$T = \{\sigma_v : v \in V\}$$

is an elementary abelian, regular subgroup of $\text{Sym}(V)$. In fact, the map

$$(2.1) \quad \begin{aligned} V &\rightarrow T \\ v &\mapsto \sigma_v \end{aligned}$$

is an isomorphism of the additive group V onto the multiplicative group T .

We consider a *key-alternating block cipher* (see Section 2.4.2 of [2]) which consists of a number of iterations of a round function of the form $\rho\sigma_k$. (Recall that we write maps left-to-right, so ρ operates first.) Here ρ is a fixed permutation operating on the vector space $V = V(n_b, 2)$, and $k \in V$ is a round key. (According to the more general definition of [2], ρ might depend on the round.) Therefore

each round consists of an application of ρ , followed by a key addition. This covers for instance AES with *independent* subkeys. Let $G = \langle \rho\sigma_k : k \in V \rangle$ the group of permutations of V generated by the round functions. Choosing $k = 0$ we see that $\rho \in G$, and thus $T \leq G$. It follows that $G = \langle T, \rho \rangle$.

3. IMPRIMITIVITY

Kenneth Paterson [12] has considered iterated block ciphers in which the group generated by the one-round functions acts imprimitively on the message space, with the aim of exploring the possibility that this might lead to the design of trapdoors. The blocks of imprimitivity he uses are the translates (cosets) of a linear subspace. He asked whether it is possible to construct other, non-linear blocks of imprimitivity:

Can “undetectable” trapdoors based on more complex systems of imprimitivity be inserted in otherwise conventional ciphers? It is easily shown that, in a DES-like cipher, any [block] system based on a linear sub-space and its cosets leads to a noticeable regularity in the XOR tables of small S-boxes. It seems that we must look beyond the “linear” systems considered here, or consider other types of round function.

In a personal communication [13], Paterson remarks further

At the FSE conference where it was presented, Adi Shamir told me that he could break the scheme using a truncated differential attack [...]

Truncated differential cryptanalysis has been introduced in [7] by L. R. Knudsen; see also the approach in [14].

In this section we answer Paterson’s question for the key-alternating block ciphers described above, by showing

Theorem 3.1. *Let G be the group generated by the round functions of a key-alternating block cipher. Suppose G acts imprimitively on the message space. Then the blocks of imprimitivity are the translates of a linear subspace.*

Proof. In the notation above, suppose G acts imprimitively on V .

If G has a nontrivial block system, this is also a block system for T . So if \mathcal{B} is a block system for G , and $B \in \mathcal{B}$ is the block containing 0, because of Lemma 2.1 we have $B = 0H$, for some $1 < H < T$. Because of the isomorphism (2.1), we have

$$H = \{ \sigma_u : u \in U \},$$

for a suitable subspace U of V , with $U \neq \{0\}, V$. Since $T = \{ \sigma_v : v \in V \}$ is abelian, we have

$$\begin{aligned} \mathcal{B} &= \{ B\sigma_v : v \in V \} = \{ 0H\sigma_v : v \in V \} = \{ 0\sigma_v H : v \in V \} = \\ &= \{ vH : v \in V \} = \{ v + U : v \in V \}. \end{aligned}$$

This completes the proof of the first implication. The converse is immediate. \square

4. TRUNCATED DIFFERENTIAL CRYPTANALYSIS

We now develop a relation to truncated differential cryptanalysis, elaborating on Shamir's comment.

Suppose G acts imprimitively on the message space V , and use the notation of the proof of Theorem 3.1. Let $v \in V$. Now $vH\rho$ is the block containing $v \cdot 1 \cdot \rho = v\rho$, so that

$$vH\rho = v\rho H,$$

for all v . This means that for all $v \in V$ and $u \in U$ there is $u' \in U$ such that

$$v\sigma_u\rho = (v + u)\rho = v\rho + u' = v\rho\sigma_{u'}.$$

In other words we have the following connection to truncated differential cryptanalysis.

Corollary 4.1. *Suppose G acts imprimitively on the message space V .*

Then there is a subspace $U \neq \{0\}$, V such that if $v, v+u \in V$ are two messages whose difference u lies in the subspace U , then the output difference also lies in U .

In other words, if $v \in V$ and $u \in U$, then

$$(4.1) \quad (v + u)\rho + v\rho \in U.$$

Conversely, if the last condition holds, then G acts imprimitively on V .

To our understanding, a subspace U as in Corollary 4.1 could indeed be used as a trapdoor as in Paterson's scheme, and still be difficult to detect. This is most clear when U is chosen to have dimension half of that of V . To a cryptanalyst who knows U , the complexity of a brute force search is reduced from $|V|$ to $2\sqrt{|V|}$. However, the number of subspaces of a given dimension m of a finite vector space of (even) dimension n over $\mathbf{GF}(2)$ is largest for $m = n/2$, and is $O(2^{m^2})$. If U is not just given by the vanishing of some of the defining bits, it appears to us that it might be hard to find. Because of this, in the next section we approach the problem of proving in a conceptual way that such a U does not exist for a given key-iterated block cipher.

5. ENSURING PRIMITIVITY

Ralph Wernsdorf has proved in [15] that the group G generated by the round functions of AES with independent subkeys is the alternating group $\text{Alt}(n)$. Thus G is definitely primitive on V .

In the following we review this consequence of Wernsdorf's result from a conceptual point of view. This comes in the form of a recipe for the group generated by the round functions of a key-alternating block cipher to be primitive. We will show that this recipe is satisfied by AES in a rather natural way.

We begin with making the description of a key-alternating block cipher we gave in Section 2 more precise. (Again, we are staying close to the notation of [2].) We assume $\rho = \gamma\lambda$, where γ and λ are permutations. Here γ is a bricklayer transformation, consisting of a number of S-boxes. The message space V is written as a direct sum

$$V = V_1 \oplus \cdots \oplus V_{n_t},$$

where each V_i has the same dimension m over $\mathbf{GF}(2)$. For $v \in V$, we will write $v = v_1 + \cdots + v_{n_t}$, where $v_i \in V_i$. Also, we consider the projections $\pi_i : V \rightarrow V_i$, which map $v \mapsto v_i$. We have

$$v\gamma = v_1\gamma_1 \oplus \cdots \oplus v_{n_t}\gamma_{n_t},$$

where the γ_i are S-boxes, which we allow to be different for each V_i .

λ is a linear mixing layer.

In AES the S-boxes are all equal, and consist of inversion in the field $\mathbf{GF}(2^8)$ with 2^8 elements (see later in this paragraph), followed by an affine transformation. The latter map thus consists of a linear transformation, followed by a translation. When interpreting AES in our scheme, we take advantage of the well-known possibility of moving the linear part of the affine transformation to the linear mixing layer, and incorporating the translation in the key addition (see for instance [10]). Thus in our scheme for AES we have $m = 8$, we identify each V_i with $\mathbf{GF}(2^8)$, and we take $x\gamma_i = x^{2^8-2}$, so that γ_i maps nonzero elements to their inverses, and zero to zero. As usual, we abuse notation and write $x\gamma_i = x^{-1}$. Note, however, that with this convention $xx^{-1} = 1$ only for $x \neq 0$.

Our result, for a key-alternating block cipher as described earlier in this section, is the following.

Theorem 5.1. *Suppose the following hold:*

- (1) $0\gamma = 0$ and $\gamma^2 = 1$, the identity transformation.
- (2) There is $1 \leq r < m/2$ such that for all i
 - for all $0 \neq v \in V_i$, the image of the map $V_i \rightarrow V_i$, which maps $x \mapsto (x + v)\gamma_i + x\gamma_i$, has size greater than 2^{m-r-1} , and
 - there is no subspace of V_i , invariant under γ_i , of codimension less than or equal to $2r$.
- (3) No sum of some of the V_i (except $\{0\}$ and V) is invariant under λ .

Then G is primitive.

We note immediately

Lemma 5.2. *AES satisfies the hypotheses of Theorem 5.1.*

Corollary 5.3. *The group generated by the round functions of AES with independent subkeys is primitive.*

Proof of Lemma 5.2. Condition (1) is clearly satisfied.

So is (3), by the construction of the mixing layer. In fact, suppose $U \neq \{0\}$ is a subspace of V which is invariant under λ . Suppose, without loss of generality, that $U \supseteq V_1$. Because of **MixColumns** [2, 3.4.3], U contains the whole first column of the state. Now the action of **ShiftRows** [2, 3.4.2] and **MixColumns** on the first column shows that U contains four whole columns, and considering (if the state has more than four columns) once more the action of **ShiftRows** and **MixColumns** one sees that $U = V$.

The first part of Condition (2) is also well-known to be satisfied, with $r = 1$ (see [11] but also [3]). We recall the short proof for convenience. For $a \neq 0$, the

map $\mathbf{GF}(2^8) \rightarrow \mathbf{GF}(2^8)$, which maps $x \mapsto (x+a)^{-1} + x^{-1}$, has image of size $2^7 - 1$. In fact, if $b \neq a^{-1}$, the equation

$$(5.1) \quad (x+a)^{-1} + x^{-1} = b$$

has at most two solutions. Clearly $x = 0, a$ are not solutions, so we can multiply by $x(x+a)$ obtaining the equation

$$(5.2) \quad x^2 + ax + ab^{-1} = 0,$$

which has at most two solutions. If $b = a^{-1}$, equation (5.1) has four solutions. Two of them are $x = 0, a$. Two more come from (5.2), which becomes

$$x^2 + ax + a^2 = a^2 \cdot ((x/a)^2 + x/a + 1) = 0.$$

By Lemma 2.2, $\mathbf{GF}(2^8)$ contains $\mathbf{GF}(4) = \{0, 1, c, c^2\}$, where c, c^2 are the roots of $y^2 + y + 1 = 0$. Thus when $b = a^{-1}$ equation (5.1) has the four solutions $0, a, ac, ac^2$. It follows that the image of the map $x \mapsto (x+a)^{-1} + x^{-1}$ has size

$$\frac{2^8 - 4}{2} + \frac{4}{4} = 2^7 - 1,$$

as claimed.

As to the second part of Condition (2), one could just use GAP [4] to verify that the only nonzero subspaces of $\mathbf{GF}(2^8)$ which are invariant under inversion are the subfields. According to Lemma 2.2, the largest proper one is thus $\mathbf{GF}(2^4)$, of codimension $4 > 2 = 2r$. However, this follows from the more general Theorem 6.1, which we give in the Appendix. \square

Proof of Theorem 5.1. Suppose, by way of contradiction, that G is imprimitive. According to Corollary 4.1, there is a subspace $U \neq \{0\}, V$ of V such that if $v, v+u \in V$ are two messages whose difference u lies in the subspace U , then the output difference also lies in U , that is

$$(v+u)\rho + v\rho \in U.$$

Since λ is linear, we have

Fact 1. For all $u \in U$ and $v \in V$ we have

$$(5.3) \quad (v+u)\gamma + v\gamma \in U\lambda^{-1} = W,$$

where W is also a linear subspace of V , with $\dim(W) = \dim(U)$.

Setting $v = 0$ in (5.3), and because of Condition (1), we obtain

Fact 2. $U\gamma = W$ and $W\gamma = U$.

Now if $U \neq \{0\}$, we will have $U\pi_i \neq \{0\}$ for some i . We prove some increasingly stronger facts under this hypothesis.

Fact 3. Suppose $U\pi_i \neq \{0\}$ for some i . Then $W \cap V_i \neq \{0\}$.

Let $u \in U$, with $u_i \neq 0$. Take any $0 \neq v_i \in V_i$. Then $(u + v_i)\gamma + v_i\gamma \in W$, and also $u\gamma \in W$, by Fact 2. It follows that $u\gamma + (u + v_i)\gamma + v_i\gamma \in W$. The latter vector has all nonzero components but for the one in V_i , which is $u_i\gamma_i + (u_i + v_i)\gamma_i + v_i\gamma_i \in W \cap V_i$. If the latter vector is zero for all $v_i \in V_i$, then the image of the map $V_i \rightarrow V_i$, which maps $v_i \mapsto (v_i + u_i)\gamma_i + v_i\gamma_i$, is $\{u_i\gamma_i\}$, of size 1. This contradicts the first part of Condition (2).

Clearly $(W \cap V_i)\gamma = U \cap V_i$. It follows

Fact 4. Suppose $U\pi_i \neq \{0\}$ for some i . Then $U \cap V_i \neq \{0\}$.

Finally we obtain

Fact 5. Suppose $U\pi_i \neq \{0\}$ for some i . Then $U \supseteq V_i$.

According to Fact 4, there is $0 \neq u_i \in U \cap V_i$. By the first part of Condition (2) the map $V_i \rightarrow V_i$, which maps $x \mapsto (x + u_i)\gamma_i + x\gamma_i$, has image of size $> 2^{m-r-1}$. Since this image is contained in the linear subspace $W \cap V_i$, it follows that the latter has size at least 2^{m-r} , that is, codimension at most r in V_i . The same holds for $U \cap V_i = (W \cap V_i)\gamma$. Thus the linear subspace $U \cap W \cap V_i$ has codimension at most $2r$ in V_i . In particular, it is different from $\{0\}$, as $m > 2r$. From Fact 2 it follows that $U \cap W \cap V_i$ is invariant under γ . By the second part of Condition (2) we have $U \cap W \cap V_i = V_i$, so that $U \supseteq V_i$ as claimed.

From Fact 5 we obtain immediately

Fact 6. U is a direct sum of some of the V_i , and $W = U$

The second part follows from the fact that $W = U\gamma$, and $V_i\gamma = V_i$ for all i .

Since $U = W\lambda$ by (5.3), we obtain $U = U\lambda$, with $U \neq \{0\}, V$. This contradicts Condition (3), and completes the proof. \square

The proof of Theorem 5.1 can be adapted to prove a slightly more general statement, in which Conditions (1) and (2) are replaced with

(1') $0\gamma = 0$ and $\gamma^s = 1$, for some $s > 1$.

(2') There is $1 \leq r < m/s$ such that for all i

- for all $0 \neq v \in V_i$, the image of the map $V_i \rightarrow V_i$, which maps $x \mapsto (x + v)\gamma_i + x\gamma_i$, has size greater than 2^{m-r-1} , and
- there is no proper subspace of V_i , invariant under γ_i , of codimension less than or equal to sr .

6. APPENDIX

We are grateful to Sandro Mattarei (see [9], and also [5], for more general results) for the following

Theorem 6.1. *Let F be a field of characteristic two. Suppose $U \neq 0$ is an additive subgroup of F which contains the inverses of each of its nonzero elements. Then U is a subfield of F .*

Proof. Hua's identity, valid in any associative (but not necessarily commutative) ring A , shows

$$(6.1) \quad a + ((a - b^{-1})^{-1} - a^{-1})^{-1} = aba$$

for $a, b \in A$, with $a, b, ab - 1$ invertible.

First of all, $1 \in U$. This is because U has even order, and each element different from 0, 1 is distinct from its inverse.

Now (6.1) for $b = 1$, and $a \in U \setminus \{0, 1\}$ shows that for $a \in U$, also $a^2 \in U$. (This is clearly valid also for $a = 0, 1$.) It follows that any $c \in U$ can be represented in the form $c = a^2$ for some $a \in U$. Now (6.1) shows that U is closed under products, so that U is a subring, and thus a subfield, of F . \square

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